Inverse scattering theory for the perturbed 1-soliton potential of the heat equation *

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Abstract

Inverse scattering transform method of the heat equation is developed for a special subclass of potentials nondecaying at space infinity—perturbations of the one-soliton potential by means of decaying two-dimensional functions. Extended resolvent, Green's functions, and Jost solutions are introduced and their properties are investigated in detail. The singularity structure of the spectral data is given and then the Inverse problem is formulated in an exact distributional sense.

1 Introduction

The equation of the heat conduction, or heat equation for short,

$$\mathcal{L}\Phi(x) = 0, \tag{1.1}$$

where the operator

$$\mathcal{L}(x, \partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u(x), \qquad x = (x_1, x_2) \tag{1.2}$$

for more than 25 years has been known [1, 2] to be associated to the Kadomtsev–Petviashvili (more precisely, KPII) equation

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = -3u_{x_2x_2}. \tag{1.3}$$

The scattering theory for the equation of heat conduction with a real potential $u(x)$ was developed in [3]–[5], but only the case of potentials rapidly decaying at large distances on the x-plane was considered. On the other side, it is well known that (1.3) is a (2+1)-dimensional generalization of the famous KdV equation: if the function $u_1(t, x_1)$ obeys KdV then

$$u(t, x_1, x_2) = u_1(t, x_1 + \mu x_2 - 3\mu^2 t) \tag{1.4}$$

solves (1.3) for an arbitrary constant $\mu \in \mathbb{R}$. Thus it is natural to consider solutions of (1.3) that are not decaying in all directions at space infinity but have 1-dimensional rays with

behavior of the type (1.4). The scattering theory for the operator (1.2) with such potentials is absent in the literature. Moreover, it is easy to observe that like in the KPI case (see [6]) the standard integral equation for the Jost solution [3] is meaningless for this situation and does not determine the solution itself. In trying to solve this problem for the nonstationary Schrödinger operator, associated to the KPI equation, a new general approach to the inverse scattering theory was introduced, which was called resolvent approach, see [6]–[10] and references therein. In [11] we developed the scattering theory for the N-soliton solutions given in terms of Bäcklund transformations of the decaying background potential. These results for the simplest case $N = 1$ are essentially used below.

Here we apply the resolvent approach to the heat equation (1.2) with a potential $u(x)$ that is a perturbation of a 1-dimensional potential $u_1(x)$ of the kind (1.4) by means of a potential $u_2(x)$ rapidly decaying in all directions

$$u(x) = u_1(x) + u_2(x). \tag{1.5}$$

We introduce and study properties of the resolvent, dressing operators, Jost solutions and scattering data and formulate the inverse problem relevant to this case. In fact we consider here the simplified version of (1.4) in which $\mu = 0$. The generic case is reconstructed by means of the Galilean invariance of (1.3). Thus in what follows $u_1(x) \equiv u_1(x_1)$ and, moreover, we consider for simplicity the case where u_1 is the 1-dimensional soliton potential (see (3.1) below).

Thus here we apply the Inverse scattering theory to a non-scattering situation since the "obstacle" is infinite. Such extension of the Inverse scattering theory results in the new and unexpected properties of familiar objects, like the Jost solutions and the spectral data. We show that they get specific singularities in the complex domain of the spectral parameter. Derivation and description of these singularities are our main results here. The article is organized as follows. In Sec. 2 we sketch some general aspects of the resolvent approach that are necessary for our construction. In Sec. 3 we present results of embedding the theory of the one-dimensional one-soliton potential in two dimensions. Presentation here is based on the work [11]. We describe in detail properties of the extended resolvent and Green's functions of operator (1.2) with $u(x) = u_1(x_1)$. On this basis in Sec. 4 the resolvent of the operator (1.2) now with the generic potential $u(x)$ given in (1.5) is introduced and its properties are described. The departure from analyticity of the resolvent leads us to definitions of the Jost solutions and spectral data and description of their properties (Sec. 5). In this way we supply all terms of the Inverse problem with proper meaning in terms of distributions. In Conclusion some generalizations and future developments of these results are discussed. Main results of this article were announced in our work [12].

2 Extension of differential operators and resolvent

In the framework of the resolvent approach we work in the space $\mathcal{S}'$ of tempered distributions $A(x, x'; q)$ of the six real variables $x = (x_1, x_2), x', q \in \mathbb{R}^2$. It is convenient to consider q as the imaginary part of a two-dimensional complex variable $\mathbf{q} = \mathbf{q}_\Re + i\mathbf{q}_\Im = (\mathbf{q}_1, \mathbf{q}_2) \in \mathbb{C}^2$ and to introduce the "shifted" Fourier transform

$$A(p; \mathbf{q}) = \frac{1}{(2\pi)^2} \int dx \int dx' \, e^{i(p+\mathbf{q}_\Re)x - i\mathbf{q}_\Re x'} A(x, x'; \mathbf{q}_\Im) \tag{2.1}$$

where $p \in \mathbb{R}^2$, $px = p_1 x_1 + p_2 x_2$ and $\mathbf{q}_\Re x = \mathbf{q}_{1\Re} x_1 + \mathbf{q}_{2\Re} x_2$. We consider the distributions $A(x, x'; q)$ and $A(p; \mathbf{q})$ as kernels in two different representations, the x- and p-representation,

respectively, of the operator $A(q)$ (A for short). The composition law in the x-representation is defined in the standard way, that is

$$(AB)(x, x'; q) = \int dx'' \, A(x, x''; q) \, B(x'', x'; q). \tag{2.2}$$

Since the kernels are distributions this composition is neither necessarily defined for all pairs of operators nor associative. In terms of the p-representation (2.1) this composition law is given by a sort of a "shifted" convolution, $(AB)(p; \mathbf{q}) = \int dp' A(p - p'; \mathbf{q} + p') B(p'; \mathbf{q})$. On the space of these operators we define the conjugation A^*, that in the x-representations reads as

$$A^*(x, x'; q) = \overline{A(x, x'; q)}, \tag{2.3}$$

where bar denotes complex conjugation, or as $A^*(p; \mathbf{q}) = \overline{A(-p; -\mathbf{q})}$ in the p-representation. Below we say that the operator $A(q)$ is real if $A^*(q) = A(q)$, that in terms of p-representation means that $\overline{A(p; \mathbf{q})} = A(-p; -\mathbf{q})$. The set of differential operators $\mathcal{D}(x, \partial_x) = \sum d_n(x) \partial_x^n$ is embedded in the introduced space of operators by considering the operators D with kernel $D(x, x') = \mathcal{D}(x, \partial_x)\delta(x - x')$, where $\delta(x) = \delta(x_1)\delta(x_2)$ is the two-dimensional δ-function and, then, by mapping them in the operators $D(q)$ with kernel

$$D(x, x'; q) \equiv e^{-q(x - x')} D(x, x') = \mathcal{D}(x, \partial_x + q)\delta(x - x'), \tag{2.4}$$

to which we refer as the **extended** version of the differential operator $\mathcal{D}$. The notion of reality for a differential operator D is exactly the condition that its coefficients $d_n(x)$ are real.

For the operator (1.2) the extension $L(q)$ is given by

$$L = L_0 - U, \tag{2.5}$$

where L_0 is the extension of $\mathcal{L}(x, \partial_x)$ in the case of zero potential, i.e. it has kernels

$$L_0(x, x'; q) = \left[-(\partial_{x_2} + q_2) + (\partial_{x_1} + q_1)^2 \right] \delta(x - x'), \qquad L_0(p; \mathbf{q}) = (i\mathbf{q}_2 - \mathbf{q}_1^2)\delta(p), \quad (2.6)$$

and the multiplication operator U can be called the potential since it has kernel

$$U(x, x'; q) = u(x)\delta(x - x'). \tag{2.7}$$

Below we always suppose that $u(x)$ is real, which by (2.3) means that the operator (1.2) is real also: $L^* = L$.

The main object of our approach is the **extended resolvent** $M(q)$ of the operator $L(q)$, which is defined as the inverse of the operator L, that is

$$LM = ML = I, \tag{2.8}$$

in the space of operators. Here I is the unity operator, $I(x, x'; q) = \delta(x - x')$, $I(p; \mathbf{q}) = \delta(p)$. In order to make this inversion uniquely defined we impose the condition that the product

$$\int dp' M(p - p'; \mathbf{q} + \mathbf{s} + p') M(p'; \mathbf{q}) \tag{2.9}$$

exists as distribution in p and $\mathbf{q}$ and that it is a continuous function of $\mathbf{s}$ in a neighborhood of $\mathbf{s} = 0$ when $\mathbf{s} \neq 0$.

Thanks to definitions (2.5), (2.6), and (2.8) M is real and in particular, the resolvent M_0 of the bare operator L_0 has in the p-representation kernel $M_0(p; \mathbf{q}) = \delta(p)(i\mathbf{q}_2 - \mathbf{q}_1^2)^{-1}$. As function of $\mathbf{q}$ it is singular when $\mathbf{q} = \ell(\mathbf{q}_1)$, where the special two-component vector

$$\ell(k) = (k, -ik^2) \qquad (2.10)$$

was introduced. In the x-representation by inverting (2.1) we get

$$M_0(x, x'; q) = \frac{1}{2\pi} \int d\alpha \left[\theta(q_1^2 - \alpha^2 - q_2) - \theta(x_2 - x_2')\right] e^{-(i\ell(\alpha+iq_1)+q)(x-x')}. \qquad (2.11)$$

For a generic operator A with kernel $A(x, x'; q)$ the operation inverse to the extension procedure, defined in (2.4) for a differential operator, is given by

$$\widehat{A}(x, x'; q) = e^{q(x-x')} A(x, x'; q). \qquad (2.12)$$

In contrast with the case of the extended differential operators for which $\widehat{D}(x, x'; q) = D(x, x') \equiv \mathcal{D}(x, \partial_x)\delta(x - x')$, in general $\widehat{A}(x, x'; q)$ does depend on q and, moreover, can have an exponential growth at space infinity. Therefore $\widehat{A}(x, x'; q)$ not necessarily belongs to the space $\mathcal{S}'$ of tempered distributions. The fact that $\widehat{A}(x, x'; q)$ can depend on q will play a crucial role in the following. For instance also in the case of the simplest resolvent (2.11) we have that the function $\widehat{M}_0(x, x'; q)$ depends effectively on the variable q and is exponentially growing at space infinity. More generally from (2.8) we have

$$\mathcal{L}(x, \partial_x)\widehat{M}(x, x'; q) = \mathcal{L}^{\mathrm{d}}(x', \partial_{x'})\widehat{M}(x, x'; q) = \delta(x - x'), \qquad (2.13)$$

where $\mathcal{L}^{\mathrm{d}}$ is the operator dual to $\mathcal{L}$. The function $\widehat{M}(x, x'; q)$ can be considered a parametric ($q \in \mathbb{R}^2$) family of Green's functions of the operator $\mathcal{L}$. In what follows we use special notations for the equalities of the type (2.13), writing them as

$$\overrightarrow{\mathcal{L}}\widehat{M}(q) = \widehat{M}(q)\overleftarrow{\mathcal{L}} = I, \qquad (2.14)$$

where $\overrightarrow{\mathcal{L}}$ denotes the operator $\mathcal{L}$ applied to the x-variable of the function $\widehat{M}(x, x'; q)$ and $\overleftarrow{\mathcal{L}}$ denotes the operator dual to $\mathcal{L}$ applied to the x'-variable of the same function. Operation (2.12) has no analog in terms of the p-representation. Nevertheless, local properties of the kernels in the x-representation are preserved, and we use the kernels with the hat in what follows intensively.

Thanks to our definitions (2.1) and (2.4) it is easy to see that in terms of the p-representation the dependence on the $\mathbf{q}$-variables of the kernels of the extension of a differential operator is polynomial (like in the example (2.6)). Correspondingly, the essential role in the study of the properties of the resolvent is played by the investigation of its departure from analyticity, in particular, by its d-bar derivatives with respect to the $\mathbf{q}$-variables. Thus to a generic operator A with kernel $A(p; \mathbf{q})$ in the p-representation we associate two operators $\bar{\partial}_j A$ with kernels

$$(\bar{\partial}_j A)(p; \mathbf{q}) = \frac{\partial A(p; \mathbf{q})}{\partial \overline{\mathbf{q}}_j}, \quad j = 1, 2, \qquad (2.15)$$

where the derivatives are considered in the sense of distributions. In terms of the objects introduced in (2.12) we get by inversion of (2.1) that

$$(\widehat{\bar{\partial}_j A})(q) = \frac{i}{2}\frac{\partial \widehat{A}(q)}{\partial q_j}. \qquad (2.16)$$

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Multiplying equalities in (2.8) from the left and right, correspondingly, by M_0 we get thanks to (2.5)

$$M = M_0 + M_0 U M, \qquad M = M_0 + M U M_0. \tag{2.17}$$

Since the resolvent M_0 is explicitly given these are integral equations determining the solution M of (2.8). In the literature (see, say, [4] and [5]) on the Jost solutions of the heat equation some small norm conditions on the potential u are known to guarantee the existence of the Jost solutions. So it is natural to assume that under such conditions the solution M of the above integral equations exists and is unique (the same for both integral equations). In this case the resolvent M can be considered as a small perturbation of the resolvent M_0 and this bare resolvent determines the properties of M by means of (2.17). The main problem of construction of the Inverse scattering transform for the operator (1.2) is that the potential $u(x)$ in (1.5) does not obey any small norm condition.

In order to overcome this difficulty we use a so called Inverse scattering transform on a non trivial background [10]. Let us consider a kind of Hilbert identity, known in the standard spectral theory of operators. Precisely, if $M(q)$ is the extended resolvent of the operator $L(q)$ with potential u and $M'(q)$ the extended resolvent of the operator $L'(q)$ with a different potential u', then, by (2.8) we have

$$M' - M = -M'(L' - L)M. \tag{2.18}$$

Strictly speaking, this follows under the assumption that the product in the r.h.s. is associative. This is a natural assumption since $L'(x, x'; q) - L(x, x'; q) = (u(x) - u(x'))\delta(x - x')$ and M satisfies condition (2.9). Let now $\mathcal{L}_1$ denote the operator (1.2) in the special case where the potential $u(x)$ in (1.5) is purely 1-dimensional, i.e. $u_2(x) \equiv 0$. Let L_1 denote its extension and M_1 its resolvent, that is let (cf. (2.5))

$$L_1 = L_0 - U_1, \qquad \mathcal{L}_1(x, \partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u_1(x), \tag{2.19}$$

$$L = L_1 - U_2, \qquad L_1 M_1 = M_1 L_1 = I, \tag{2.20}$$

where as in (2.7) $U_j(x, x'; q) = u_j(x)\delta(x - x')$. Choosing now in (2.18) $L' = L_1$ we get

$$M = M_1 + M_1 U_2 M, \quad M = M_1 + M U_2 M_1, \tag{2.21}$$

where the second equality is derived in analogy. These equations generalize (2.17) for the case where M_1 is non trivial and if the resolvent M_1 is known they also can be considered as defining the resolvent M. If we choose U_2 obeying the small norm condition mentioned above, we can assume that the solution M of both equations (2.21) exists and is unique. Then, thanks to (2.20) M obeys (2.8). Contrary to (2.17) now M can be considered a perturbation of the resolvent M_1. So in the next section we study the properties of the resolvent M_1 in detail.

3   One-dimensional potential

We already mentioned in the Introduction that in this article we deal with the case where u_1 in (1.5) is the 1-dimensional soliton potential

$$u_1(x) = \frac{-2a^2}{\cosh^2 [a (x_1 - x_0)]}, \tag{3.1}$$

with $a > 0$ and x_0 real constants. In this section we consider the case where the perturbation is absent, $u_2 \equiv 0$. We re-formulate in the 2-dimensional space the well known facts about Jost solutions of this 1-soliton potential and introduce and study the properties of the resolvent and Green's functions in this case.

The differential equations $\mathcal{L}_1(x, \partial_x)\Phi_1(x, k) = 0$, $\mathcal{L}_1^{\mathrm{d}}(x, \partial_x)\Psi_1(x, k) = 0$ for the Jost solution $\Phi_1(x, k)$ and its dual $\Psi_1(x, k)$, by using the notation introduced in (2.14), can be re-written as follows

$$\overrightarrow{\mathcal{L}}_1 \Phi_1(k) = 0, \quad \Psi_1(k)\overleftarrow{\mathcal{L}}_1 = 0. \tag{3.2}$$

Here and in the following we omit the x-dependence when it is irrelevant. These solutions are given explicitly by

$$\Phi_1(x, k) = \frac{k - ia \tanh\left[a(x_1 - x_0)\right]}{k - ia} e^{-i\ell(k)x}, \tag{3.3}$$

$$\Psi_1(x, k) = \frac{k + ia \tanh\left[a(x_1 - x_0)\right]}{k + ia} e^{i\ell(k)x}, \tag{3.4}$$

where $k \in \mathbb{C}$ and the two-component vector $\ell(k)$ is defined in (2.10). They obey the conjugation properties

$$\overline{\Phi_1(x, k)} = \Phi_1(x, -\bar{k}), \quad \overline{\Psi_1(x, k)} = \Psi_1(x, -\bar{k}) \tag{3.5}$$

that are equivalent to the reality condition for the potential u_1, and are normalized at k-infinity as follows

$$\lim_{k \to \infty} e^{i\ell(k)x}\Phi_1(x, k) = 1, \qquad \lim_{k \to \infty} e^{-i\ell(k)x}\Psi_1(x, k) = 1. \tag{3.6}$$

The functions $\Phi_1(x, k)$ and $\Psi_1(x, k)$ are meromorphic in the complex domain of the spectral parameter k with poles at $k = ia$ and $k = -ia$, correspondingly. Thus, these functions obey the d-bar equations

$$\frac{\partial \Phi_1(x, k)}{\partial \bar{k}} = i\pi \Phi_{1,a}(x)\delta(k - ia), \qquad \frac{\partial \Psi_1(x, k)}{\partial \bar{k}} = i\pi \Psi_{1,-a}(x)\delta(k + ia), \tag{3.7}$$

where we introduced the notations

$$\Phi_{1,a}(x) = -i \operatorname*{res}_{k=ia} \Phi_1(x, k), \qquad \Psi_{1,-a}(x) = -i \operatorname*{res}_{k=-ia} \Psi_1(x, k). \tag{3.8}$$

Explicitly we have

$$\Phi_{1,a}(x) = \frac{ae^{ax_0 + a^2 x_2}}{\cosh\left[a(x_1 - x_0)\right]}, \quad \Psi_{1,-a}(x) = -\frac{ae^{ax_0 - a^2 x_2}}{\cosh\left[a(x_1 - x_0)\right]}. \tag{3.9}$$

Let

$$c = 2ae^{2ax_0} \tag{3.10}$$

and $\Phi_{1,-a}(x)$ and $\Psi_{1,a}(x)$ the values of the Jost solutions in the conjugated points,

$$\Phi_{1,-a}(x) = \Phi_1(x, -ia), \qquad \Psi_{1,a}(x) = \Psi_1(x, ia). \tag{3.11}$$

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Then thanks to (3.3), (3.4), and (3.9) the Jost solutions obey in the complex domain of the spectral parameter the following scalar products

$$\int dx_1 \Psi_1(x, k+p)\Phi_1(x, k) = 2\pi\delta(p), \quad p \in \mathbb{R}, \tag{3.12}$$

$$c \int dx_1 \, \Phi_{1,-a}(x)\Psi_{1,a}(x) = 1, \tag{3.13}$$

$$\int dx_1 \, \Psi_{1,a}(x)\Phi_1(x, k) = 0, \quad k_\Im^2 < a^2, \tag{3.14}$$

$$\int dx_1 \, \Phi_{1,-a}(x)\Psi_1(x, k) = 0, \quad k_\Im^2 < a^2, \tag{3.15}$$

and the completeness relation

$$\frac{1}{2\pi} \int\limits_{x_2=x_2'} dk_\Re \, \Phi_1(x, k)\Psi_1(x', k) + c\theta(a^2 - k_\Im^2)\Phi_{1,-a}(x)\Psi_{1,a}(x')\Big|_{x_2=x_2'} = \delta(x_1 - x_1'). \tag{3.16}$$

Equations (3.7) can be considered as two Inverse problems defining the Jost solution and the dual Jost solution. The formulation of these problems is closed by giving the normalization conditions (3.6) and the following relations:

$$\Phi_{1,a}(x) = c\Phi_{1,-a}(x), \qquad \Psi_{1,-a}(x) = -c\Psi_{1,a}(x), \tag{3.17}$$

where $\Phi_{1,-a}(x)$ and $\Psi_{1,a}(x)$ are defined in (3.11).

These formulae show that the embedding in two dimensions of the Jost solutions of the 1-soliton potential is trivial and just mimics the 1-dimensional construction up to the warning that, due to their exponential dependence on x_2, the functions $\Phi_{1,a}(x)$ and $\Psi_{1,-a}(x)$ are not square integrable with respect to the x-variables and, therefore, are not eigenfunctions of the operator $\mathcal{L}_1$.

On the contrary, the 2-dimensional resolvent $M_1(q)$ as well as the Green's function G_1 of the 2-dimensional operator $\mathcal{L}_1$ with the 1-dimensional potential u_1 are not trivial extensions of the corresponding 1-dimensional objects associated to the operator $(k^2 + \partial_{x_1}^2 - u_1(x_1))$. In terms of the Jost solutions introduced above we can write the kernel of this resolvent obtained in [11] as

$$\widehat{M_1}(x, x'; q) = \frac{1}{2\pi} \int\limits_{k_\Im=q_1} dk_\Re \left[\theta(q_1^2 - q_2 - k_\Re^2) - \theta(x_2 - x_2')\right] \Phi_1(x, k)\Psi_1(x', k) +$$
$$+ c\theta(a^2 - q_1^2)\left[\theta(a^2 - q_2) - \theta(x_2 - x_2')\right]\Phi_{1,-a}(x)\Psi_{1,a}(x') \tag{3.18}$$

where the hat over the kernel is used in the sense of notation (2.12).

Thanks to the equalities (3.2) and (3.16) it is easy to check directly that $\widehat{M_1}(q)$ obeys the equations

$$\overrightarrow{\mathcal{L}}_1\widehat{M_1}(q) = \widehat{M_1}(q)\overleftarrow{\mathcal{L}}_1 = I, \tag{3.19}$$

that means (cf. (2.14)) that $M_1(q)$ obeys (2.20) and is indeed the inverse of the operator $L_1(q)$. Moreover, using the explicit formulas (3.3), (3.4), and (3.9) we get that $M_1(x, x'; q) \in \mathcal{S}'(R^6)$, i.e. it belongs to the space of operators under consideration. It can also be proved directly that M_1 obeys condition (2.9), so it is the extended resolvent according to our definition. By means of (3.5) we get also that M_1 is real, $M_1^* = M_1$ according to definition (2.3).

We emphasize that in order to prove these results it is not necessary to use the explicit formulae for Φ_1 and Ψ_1 but only their general properties. In fact, if one considers a 1-dimensional potential u_1 which, in addition, has a non trivial continuous spectrum one gets the same formula for the resolvent M_1. If the discrete (1-dimensional) spectrum of u_1 contains N solitons with parameters a_j and c_j ($j = 1, 2, \ldots, N$), then the last term in (3.18) must be substituted by the sum of similar terms each corresponding to a value of j.

Now we describe in detail the properties of $M_1(x.x'; q)$. The first term in the r.h.s. of (3.18) is a continuous function of $q = (q_1, q_2)$ on the q-plane with discontinuities on the lines $q_1 = \pm a$ due to the pole singularities of $\Phi_1(k)$ and $\Psi_1(k)$. The second term thanks to the θ functions has discontinuities on the lines $q_1 = \pm a$ and on the cut $q_2 = a^2$, $|q_1| < a$. The singularities on the lines $q_1 = \pm a$ are exactly compensated among the two terms. Thus the kernel $M_1(x, x'; q)$ is a continuous function of q with a discontinuity on the cut $q_2 = a^2$, $|q_1| < a$. This discontinuity is specific of the potential $u_1(x)$, or more generally of a potential with discrete spectrum and it gives the essential difference of M_1 with respect to the bare resolvent M_0 (2.11). Let us underline that, in spite of the fact that $\mathcal{L}_1$ applied to the term with $\theta(a^2 - q_2)$ that causes this discontinuity gives zero, this term cannot be omitted in (3.18), since only thanks to the fact that $\theta(x_2 - x_2')$ and $\theta(a^2 - q_2)$ have opposite signs the kernel $M_1(x, x'; q) \equiv e^{-q(x-x')}\widehat{M_1}(x, x'; q)$ is a tempered distribution with respect to the x-variables.

The kernel $M_1(p; \mathbf{q})$ in the p-representation is not an analytic function of $\mathbf{q}$. By (2.16) the d-bar derivatives of M_1 with respect to $\mathbf{q}_j$ are proportional to $\partial \widehat{M_1}/\partial q_j$ and for the latter we get from (3.18) equalities

$$\frac{\partial \widehat{M_1}(q)}{\partial q_1} = \frac{i}{\pi} \int\limits_{k_\Im = q_1} dk_\Re \, \bar{k}\delta(\ell_{2\Im}(k) - q_2)\, \Phi_1(k) \otimes \Psi_1(k), \qquad (3.20)$$

$$\frac{\partial \widehat{M_1}(q)}{\partial q_2} = \frac{-1}{2\pi} \int\limits_{k_\Im = q_1} dk_\Re \, \delta(\ell_{2\Im}(k) - q_2)\, \Phi_1(k) \otimes \Psi_1(k), \qquad (3.21)$$

where $(\Phi_1(k) \otimes \Psi_1(k))(x, x') \equiv \Phi_1(x, k)\Psi_1(x', k)$ is the standard direct product and where by the above discussion we consider $q_2 \neq a^2$. For the discontinuity along this line we get

$$\widehat{M_1}(q)\Big|_{q_2 = a^2 + 0} - \widehat{M_1}(q)\Big|_{q_2 = a^2 - 0} = -c\theta(a^2 - q_1^2)\Phi_{1,-a} \otimes \Psi_{1,a}. \qquad (3.22)$$

We have to study now the behavior of $M_1(q)$ at the end points of the cut, i.e. when $q \sim (\pm a, a^2)$. Firstly, it is convenient to subtract to $M_1(q)$ its value, say, on the upper or lower edges of the cut:

$$g_1^\pm = \lim_{q_2 = a^2 \pm 0} \widehat{M_1}(q)\Big|_{|q_1| < a}. \qquad (3.23)$$

Since $\Phi_1(k)$ and $\Psi_1(k)$ are analytic for $|k_\Im| < a$, we deduce from (3.18) that $g_1^\pm$ are independent also of q_1 and their kernels equal

$$g_1^\pm(x, x') = -\frac{\theta(x_2 - x_2')}{2\pi} \int d\alpha \, \Phi_1(x, \alpha)\Psi_1(x', \alpha) \mp c\theta(\pm(x_2 - x_2'))\Phi_{1,-a}(x)\Psi_{1,a}(x'),$$

$$(3.24)$$

where $\int d\alpha$ denotes integration along the whole real axis. Now extracting explicitly from the first term in the r.h.s. of (3.18) the contribution coming from the poles of $\Phi_1(k)$ and

$\Psi_1(k)$ we get that, say, difference $\widehat{M}_1(q) - g_1^-$ behaves at points $q = (\pm a, a^2)$ as

$$\widehat{M}_1(q) - g_1^- = -c\left(\frac{\theta(q_1^2 - q_2)}{\pi}\operatorname{arccot}\frac{a - |q_1|}{\sqrt{q_1^2 - q_2}} + \theta(q_2 - q_1^2)\theta(q_2 - a^2)\right)\Phi_{1,-a} \otimes \Psi_{1,a} +$$
$$+ o(1), \quad q \sim (\pm a, a^2). \quad (3.25)$$

Thus $\widehat{M}_1(q)$ is bounded but discontinuous at $q = (\pm a, a^2)$, while its regular part, g_1^-, is the same for both these points.

Now it is easy to see that the discontinuity of the resolvent along the cut $q_2 = a^2$, $|q_1| < a$ and the ill definiteness at the points $q = (\pm a, a^2)$ are the result of embedding the 1-dimensional potential in the two dimensional space. Indeed, the resolvent of the Sturm–Liouville operator $\partial_{x_1}^2 - u_1(x_1) - q_2$ is obtained from $M_1(q)$ by means of the operation $\int dx_2\, e^{-q_2(x_2 - x_2')}\widehat{M}_1(x, x'; q)$. By (3.18) and (3.3), (3.4), (3.9) we get the standard expression for the 1-dimensional Green's function with a pole at $q_2 = a^2$.

We already noted that $\widehat{M}_1(x, x'; q)$ defines a family of Green's functions. Among them we expect should play a special role those obtained considering the values of q belonging to the support of the defects of analyticity given in (3.20), (3.21) and in (3.22). We consider, therefore, the Green's functions

$$G_1(x, x', k) = \widehat{M}_1(x, x'; q)\Big|_{q = \ell_\Im(k)}, \quad (3.26)$$

$$G_1^\pm(x, x'; k) = \widehat{M}_1(x, x'; q)\Big|_{q_1 = k_\Im,\, q_2 = a^2 \pm 0}, \quad (3.27)$$

where $k \in \mathbb{C}$ is the spectral parameter and we denote $q_1 = k_\Im$ (see (2.10)) in order to meet the standard notation. From these definitions it follows directly that

$$\overrightarrow{\mathcal{L}}_1 G_1(k) = G_1(k)\overleftarrow{\mathcal{L}}_1 = I, \qquad \overrightarrow{\mathcal{L}}_1 G_1^\pm(k) = G_1^\pm(k)\overleftarrow{\mathcal{L}}_1 = I, \quad (3.28)$$

$$\overline{G_1(k)} \equiv G_1(-\bar{k}) \equiv G_1(k), \quad (3.29)$$

$$G_1(k)\Big|_{k_\Re = 0} = G_1(0), \qquad G_1^\pm(k) = G_1^\pm(ik_\Im), \quad (3.30)$$

i.e. $G_1^\pm(k)$ are independent on $k_\Re$ and then inside the strip they coincide with $g_1^\pm$ introduced in (3.23),

$$G_1^\pm(x, x'; k)\Big|_{|k_\Im| < a} = g_1^\pm(x, x'). \quad (3.31)$$

As well from (3.18) we get the representations

$$G_1(x, x', k) =$$
$$= \frac{1}{2\pi}\int\limits_{k_\Im' = k_\Im} dk'\left[\theta(|k_\Re| - |k_\Re'|) - \theta(x_2 - x_2')\right]\Phi_1(x, k')\Psi_1(x', k') +$$
$$+ c\,\theta(a - |k_\Im|)\theta(x_2' - x_2)\Phi_{1,-a}(x)\Psi_{1,a}(x'), \quad (3.32)$$

$$G_1^\pm(x, x', k) =$$
$$= \frac{1}{2\pi}\int\limits_{k_\Im' = k_\Im} dk_\Re'\left[\theta\left((k_\Im)^2 - a^2 - (k_\Re')^2\right) - \theta(x_2 - x_2')\right]\Phi_1(x, k')\Psi_1(x', k') \mp$$
$$\mp c\,\theta(a^2 - k_\Im^2)\theta(\pm(x_2 - x_2'))\Phi_{1,-a}(x)\Psi_{1,a}(x'). \quad (3.33)$$

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The first of these equalities shows that the cut of the resolvent at $q_2 = a^2$, $|q_1| < a$ is not inherited by $G_1(k)$ (in contrast to the case of the nonstationary Schrödinger equation, as mentioned in the Introduction) and that $G_1(k)$ is discontinuous only at the points $k = \pm ia$. Its behavior in the neighborhoods of these points follows from (3.25) and reads as

$$G_1(k) = g_1^- - \frac{c}{\pi} \left\{ \mathrm{arccot}\, \frac{a - |k_\Im|}{|k_\Re|} \right\} \Phi_{1,-a} \otimes \Psi_{1,a} + o(1), \quad k \sim \pm ia. \qquad (3.34)$$

Also the Green's functions $G_1^\pm(k)$ are discontinuous only at $k_\Im = \pm a$ and one gets thanks to (3.25) that for $k \sim ia$ or $k \sim -ia$

$$G_1^\pm(k) = g_1^- - c\frac{1 \pm \theta(a - |k_\Im|)}{2} \Phi_{1,-a} \otimes \Psi_{1,a} + o(1). \qquad (3.35)$$

Notice that these functions $G_1^\pm(k)$ coincide when $|k_\Im| > a$ and are independent of $k_\Im$ (and then of k by (3.30)) when $|k_\Im| < a$. On the borders of these strips they have the discontinuity

$$G_1^\pm(k)\Big|_{\mathrm{outside}} - G_1^\pm(k)\Big|_{\mathrm{inside}} = \pm\frac{c}{2}\Phi_{1,-a} \otimes \Psi_{1,a}. \qquad (3.36)$$

Taking into account the discontinuous behavior of the Green's functions we see that equalities of the type $G_1^\pm(ia) = G_1(ia)$ and $G_1^\pm(-ia) = G_1(-ia)$ have no meaning in our case. Thanks to (3.32) and (3.35) we have only that

$$\lim_{|k_\Im| \to a-0} \lim_{k_\Re \to 0} G_1(k) = g_1^-, \qquad (3.37)$$

where the limiting procedure must be performed in such way that $|k_\Re|/(a - |k_\Im|) \to +0$.

In order to complete the study of the Green's functions we mention that $G_1(k)$ obeys the standard equalities

$$\lim_{k \to \infty} (-2ik)\frac{\partial}{\partial x_1} e^{i\ell(k)(x-x')} G_1(x, x', k) = \delta(x - x'), \qquad (3.38)$$

$$\frac{\partial G_1(x, x', k)}{\partial \bar{k}} = \frac{\mathrm{sgn}\, k_\Re}{2\pi} \Phi_1(x, -\bar{k})\Psi_1(x', -\bar{k}). \qquad (3.39)$$

The first of them follows either from the differential equations (3.28), or from the integral representation (3.32) and properties (3.6). The second one also follows from (3.28), or it can be derived from (3.26) by means of (3.20) and (3.21). This equality must be understood in the sense of distributions and we see that the discontinuity of $G_1(k)$ at points $k = \pm ia$ leads (by (3.3), (3.4)) to the pole singularities of the r.h.s. at these points. In view of (3.39) in what follows we refer to $G_1(k)$ as the Green's function of the Jost solutions.

4 Resolvent of the perturbed L-operator

Now we consider the general case of the operator (1.2) with potential given in (1.5), where $u_2(x)$ is a real function of two space variables, smooth and rapidly decaying at space infinity. The extended resolvent $M(q)$ is determined by (one of) Eqs. (2.21) and we need to study its analyticity properties first. The increment $M(p; \mathbf{q} + \mathbf{s}) - M(p; \mathbf{q})$ of M can be obtained from the Hilbert identity (2.18) where prime means the increment $\mathbf{s}$ of $\mathbf{q}$. We have $M' - M = -M'(L_1' - L_1)M$ and, then, using (2.20)

$$M' - M = M'L_1'(M_1' - M_1)L_1 M. \qquad (4.1)$$

Thus for the d-bar derivatives with respect to $\mathbf{q}_j$ we get

$$\bar{\partial}_j M = (ML_1)(\bar{\partial}_j M_1)(L_1 M), \quad j = 1, 2, \tag{4.2}$$

in the region where M_1 is continuous, i.e. for $q_2 \neq a^2$. In terms of the objects introduced in (2.12), we obtain

$$\widehat{\bar{\partial}_j M} = \widehat{M}\overleftarrow{\mathcal{L}}_1 (\widehat{\bar{\partial}_j M_1})\overrightarrow{\mathcal{L}}_1 \widehat{M}, \quad j = 1, 2, \tag{4.3}$$

where we used that $\widehat{L}_1(x, x'; q) = \mathcal{L}_1 \delta(x - x')$ and took into account that when kernels with hat are considered the multiplication by $\mathcal{L}_1$ is no more associative and it is necessary to use the arrows to indicate the correct order of operations (cf. (2.14)). Now, thanks to (2.16) and using (3.20) and (3.21) we get for $q_2 \neq a^2$

$$\frac{\partial \widehat{M}(q)}{\partial q_1} = \frac{i}{\pi} \int\limits_{k_\Im = q_1} dk_\Re \, \bar{k}\delta(\ell_{2\Im}(k) - q_2) \, \Phi(k) \otimes \Psi(k), \tag{4.4}$$

$$\frac{\partial \widehat{M}(q)}{\partial q_2} = \frac{-1}{2\pi} \int\limits_{k_\Im = q_1} dk_\Re \, \delta(\ell_{2\Im}(k) - q_2) \, \Phi(k) \otimes \Psi(k), \tag{4.5}$$

where $\Phi(k)$ and $\Psi(k)$ are defined by

$$\Phi(k) = G(k)\overleftarrow{\mathcal{L}}_1 \Phi_1(k), \quad \Psi(k) = \Psi_1(k)\overrightarrow{\mathcal{L}}_1 G(k), \tag{4.6}$$

with

$$G(x, x', k) = \widehat{M}(x, x'; q)\Big|_{q = \ell_\Im(k)}. \tag{4.7}$$

More explicitly, say, the first of equations (4.6) stands for $\Phi(x, k) = \int dx' \big(\mathcal{L}_1^d(x', \partial_{x'}) \times G_1(x, x', k)\big)\Phi_1(x', k)$. The function $G(k)$ with kernel $G(x, x', k)$ defined in (4.6) satisfies the differential equations

$$\overrightarrow{\mathcal{L}} G(k) = G(k)\overleftarrow{\mathcal{L}} = I, \tag{4.8}$$

which can be obtained as a direct reduction of (2.14). Therefore, $G(k)$ is a Green's function. Since the reduction is the same used in (3.26) for getting $G_1(k)$ from $\widehat{M}_1$ we derive from (2.21) that this Green's function obeys the integral equations

$$G(k) = G_1(k) + G_1(k)U_2 G(k), \qquad G(k) = G_1(k) + G(k)U_2 G_1(k). \tag{4.9}$$

Again, as in Sec. 3, thanks to (4.7) and (4.4), (4.5) we get the d-bar derivative of the Green's function in the form

$$\frac{\partial G(k)}{\partial \bar{k}} = \frac{\mathrm{sgn}\, k_\Re}{2\pi} \, \Phi(-\bar{k}) \otimes \Psi(-\bar{k}), \tag{4.10}$$

where $\Phi(k)$ and $\Psi(k)$ are defined in (4.6). These objects due to their definition and (4.9) obey the integral equations

$$\Phi(k) = \Phi_1(k) + G_1(k)U_2 \Phi(k), \quad \Psi(k) = \Psi_1(k) + \Psi(k)\, U_2 G_1(k), \tag{4.11}$$

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where again the first equation more explicitly reads as $\Phi(x, k) = \Phi_1(x, k) + \int dx' G_1(x, x', k) \times u_2(x')\Phi(x', k)$. It is clear that the differential equations

$$\overrightarrow{\mathcal{L}}\,\Phi(k) = 0, \qquad \Psi(k)\overleftarrow{\mathcal{L}} = 0, \tag{4.12}$$

hold and, therefore, we can consider $\Phi(x, k)$ and $\Psi(x, k)$ as the generalization of the Jost solutions to the case where the perturbation $u_2(x)$ is different from zero. Let us mention that thanks to these definitions we succeeded to avoid the indeterminacy in the definition of the Jost solutions discussed in the Introduction. Below we study the properties of the Green's function and the Jost solutions in more details and discuss the singular structure of the terms involved in (4.10). Now let us mention the following standard properties

$$-2i \lim_{k \to \infty} k \partial_{x_1} \left(e^{i\ell(k)(x-x')} G(x, x', k) \right) = \delta(x - x'), \tag{4.13}$$

$$\overline{G(k)} = G(-\bar{k}) = G(k), \tag{4.14}$$

$$\overline{\Phi(x, k)} = \Phi(x, -\bar{k}), \qquad \overline{\Psi(x, k)} = \Psi(x, -\bar{k}), \tag{4.15}$$

that can be obtained by means of the integral equations (4.9) and properties (3.5), (3.29), and (3.38) for the Green's function $G_1(k)$.

Till now we studied the departure from analyticity of the resolvent in the case $q_2 \neq a^2$. Since the resolvent $M_1(q)$ is discontinuous along the line $q_2 = a^2$ (see (3.22)), the integral equations (2.21) suggest that also $M(q)$ has a discontinuity. Let us denote the limiting values on the two edges of the line by

$$M^{\pm}(q) = M(q)\Big|_{q_2 = a^2 \pm 0}. \tag{4.16}$$

Then from the Hilbert identity (4.1) we derive that

$$M^+(q) - M^-(q) = M^{\pm}(q) L_1(q)(M_1^+(q) - M_1^-(q)) L_1(q) M^{\mp}(q), \quad q_2 = a^2, \tag{4.17}$$

where the l.h.s. is independent of the choice of the sign in the r.h.s. In analogy with (3.27) we introduce the two Green's functions

$$G^{\pm}(x, x'; k) = \widehat{M}(x, x'; q)\Big|_{q_1 = k_\Im, \, q_2 = a^2 \pm 0} \tag{4.18}$$

and rewrite (4.17) in these terms as $G^+(k) - G^-(k) = (G^{\pm}(k)\overleftarrow{\mathcal{L}}_1)(G_1^+(k) - G_1^-(k)) \times (\overrightarrow{\mathcal{L}}_1 G^{\mp}(k))$. Then by (3.22) and (3.27) we get

$$G^+(k) - G^-(k) = -c\theta(a^2 - k_\Im^2)\Phi^{\pm}(k) \otimes \Psi^{\mp}(k), \tag{4.19}$$

where the new solutions (cf. (4.6))were introduced:

$$\Phi^{\pm}(k) = G^{\pm}(k)\overleftarrow{\mathcal{L}}_1 \Phi_{1,-a}, \qquad \Psi^{\pm}(k) = \Psi_{1,a}\overrightarrow{\mathcal{L}}_1 G^{\pm}(k). \tag{4.20}$$

Following properties of $G_1^{\pm}(k)$ it is easy to show that these Green's functions obey the following differential and integral equations and reality condition

$$\overrightarrow{\mathcal{L}}\,G^{\pm}(k) = G^{\pm}(k)\overleftarrow{\mathcal{L}} = I, \tag{4.21}$$

$$G^{\pm}(k) = G_1^{\pm}(k) + G_1^{\pm}(k)U_2 G^{\pm}(k), \qquad G^{\pm}(k) = G_1^{\pm}(k) + G^{\pm}(k)U_2 G_1^{\pm}(k), \tag{4.22}$$

$$\overline{G^{\pm}(k)} = G^{\pm}(k). \tag{4.23}$$

By definition they are independent of $k_\Re$ and by the corresponding properties of $G_1^\pm(k)$ we have that $G^+(k) = G^-(k)$ when $|k_\Im| > a$ and they are independent of $k_\Im$ when $|k_\Im| < a$. By (4.20) and (4.21) we get that $\Phi^\pm(x, k)$ and $\Psi^\pm(k)$ are solutions of the heat equation with potential (1.5),

$$\overrightarrow{\mathcal{L}}\,\Phi^\pm(k) = 0, \qquad \Psi^\pm(k)\overleftarrow{\mathcal{L}} = 0. \tag{4.24}$$

Integral equations for these solutions follow by applying operations (4.20) to the equations (4.22)

$$\Phi^\pm = \Phi_{1,-a} + G_1^\pm U_2 \Phi^\pm, \qquad \Psi^\pm = \Psi_{1,a} + \Psi^\pm U_2 G_1^\pm. \tag{4.25}$$

Let us also mention that thanks to (4.23) these solutions are real and are independent of k inside the strip $|k_\Im| < a$, due to the corresponding property of $G^\pm(k)$ and (4.20). Since in the following we use intensively the Green's functions and these solutions inside the strip it is convenient to introduce the following specific notations

$$g^\pm(x, x') = G^\pm(x, x', k)\Big|_{|k_\Im|<a}, \tag{4.26}$$

and also

$$\phi^\pm(x) = \Phi^\pm(x, k)\Big|_{|k_\Im|<a}, \qquad \psi^\pm(x) = \Psi^\pm(x, k)\Big|_{|k_\Im|<a}. \tag{4.27}$$

Equality (4.19) enables us to find relations between solutions (4.20). Let $|k_\Im| < a$, then applying, say, $\overleftarrow{\mathcal{L}}_1\Phi_{1,-a}$ to this equality from the right and using (4.20) we derive that

$$(1 + \lambda)\phi^+ = \phi^-, \tag{4.28}$$

where

$$\lambda = c(\Psi_{1,a}\overrightarrow{\mathcal{L}}_1 g^-\overleftarrow{\mathcal{L}}_1\Phi_{1,-a}). \tag{4.29}$$

Explicitly $\lambda = c\int dx \int dx' \Psi_{1,a}(x)(\mathcal{L}_1(x, \partial_x)\mathcal{L}_1^{\mathrm{d}}(x', \partial_{x'})g^-(x, x')\Phi_{1,-a}(x')$. By (4.23) this constant is real and thanks to (4.20) it is also equal to $\lambda = c(\Psi_{1,a}\overrightarrow{\mathcal{L}}_1\phi^-) = c(\psi^-\overleftarrow{\mathcal{L}}_1\Phi_{1,-a})$. Inserting here $\mathcal{L}_1 = \mathcal{L} + U_2$ we get by (4.21) and (4.24) that

$$\lambda = c\{(\Psi_{1,a}U_2\Phi_{1,-a}) + (\Psi_{1,a}U_2 g^- U_2 \Phi_{1,-a})\}, \tag{4.30}$$

or $\lambda = c(\Psi_{1,a}U_2\phi^-) = c(\psi^- U_2\Phi_{1,-a})$, where we also used $\overrightarrow{\mathcal{L}}_1\Phi_{1,-a} = 0$ and $\Psi_{1,a}\overleftarrow{\mathcal{L}}_1 = 0$ that follows from (3.2) and (3.8). Next, applying to (4.28) $\Psi_{1,a}\overrightarrow{\mathcal{L}}_1$ from the left and again by (4.20) we get

$$(1 + \lambda)[1 - c(\Psi_{1,a}\overrightarrow{\mathcal{L}}_1 g^+\overleftarrow{\mathcal{L}}_1\Phi_{1,-a})] = 1, \tag{4.31}$$

where a new constant $(\Psi_{1,a}\overrightarrow{\mathcal{L}}_1 g^+\overleftarrow{\mathcal{L}}_1\Phi_{1,-a}) = (\Psi_{1,a}U_2\Phi_{1,-a}) + (\Psi_{1,a}U_2 g^+ U_2\Phi_{1,-a})$ (cf. (4.30)) appeared. Since we chose u_2 to be rapidly decaying at infinity all terms must be finite. Then $1 + \lambda \neq 0$ and, more precisely, taking into account that for $u_2 \to 0$ also $\lambda \to 0$ we have that

$$1 + \lambda > 0. \tag{4.32}$$

Summarizing, we get the following relations:

$$c(\Psi_{1,a} \overrightarrow{\mathcal{L}}_1 g^+ + \overleftarrow{\mathcal{L}}_1 \Phi_{1,-a}) = \frac{\lambda}{1+\lambda}, \tag{4.33}$$

$$\phi^+ = \frac{\phi^-}{1+\lambda}, \qquad \psi^+ = \frac{\psi^-}{1+\lambda}, \tag{4.34}$$

$$G^+(k) = G^-(k) - \frac{c\theta(a-|k_\Im|)}{1+\lambda}\phi^- \otimes \psi^-. \tag{4.35}$$

Here (4.33) is just (4.31), the first equality in (4.34) is (4.29) and the second equality is derived by analogy, and (4.35) follows from (4.19) thanks to (4.34). In their turn (4.33) and (4.34) follow from (4.35) thanks to (4.20) and (4.28).

We have shown in (3.36) that the Green's functions $G_1^\pm(k)$ are discontinuous at $k_\Im = a$ and $k_\Im = -a$. By (4.22) we deduce that $G^\pm(k)$ have the same behavior. In order to study this discontinuity we use, as above, the Hilbert identity (4.1) where $M = M(q)$, $M' = M(q')$, etc. We choose $q_2 = q_2' = a^2 \pm 0$, $q_1 = a - \varepsilon$, $q_1' = a + \varepsilon$, and in the limit $\varepsilon \to +0$ we use the hat notation (2.12) and definitions (3.27) and (4.18) of the Green's functions. Then we get $G^\pm(i(a+0)) - g^\pm = G^\pm(i(a+0))\overleftarrow{\mathcal{L}}_1(G_1^\pm(i(a+0)) - g_1^\pm)\overrightarrow{\mathcal{L}}_1 g^\pm$, where again (3.31) and (4.26) where used. Now by (3.36) for the discontinuity of the unperturbed Green's functions we obtain

$$G^\pm(i(a+0)) - g^\pm = \pm\frac{c}{2}\Phi^\pm(i(a+0)) \otimes \psi^\pm \tag{4.36}$$

where notations (4.20), (4.26) and (4.27) were used. Applying $\overleftarrow{\mathcal{L}}_1\Phi_{1,-a}$ from the right and $\Psi_{1,a}\overrightarrow{\mathcal{L}}_1$ from the left in analogy with the derivation of (4.35) we get by (4.28) that $c(\Psi_{1,a}\overrightarrow{\mathcal{L}}_1 G^\pm(i(a+0))\overleftarrow{\mathcal{L}}_1\Phi_{1,-a}) = 2\lambda(2+\lambda)^{-1}$, that is finite due to (4.32). Then omitting details we derive the equalities $G^\pm(i(a+0)) = G^\pm(-i(a+0)) = g^- - c(2+\lambda)^{-1}\phi^- \otimes \psi^-$, that, say, for the bottom sign can also be rewritten in the form

$$G^-(k) = g^- - \frac{c\theta(|k_\Im| - a)}{2+\lambda}\phi^- \otimes \psi^- + o(1), \quad k \sim \pm ia, \tag{4.37}$$

where we took (4.26) into account.

5   Properties of the Jost solutions and Inverse problem

In this section we complete the investigation of the properties of the Jost solutions by describing their behavior at the points $k = \pm ia$. Formulae (4.6) suggest to study first the behavior of the Green's function $G(k)$. We expect that it is ill defined at these points, so in order to describe this behavior we compare $G(k)$ with some well defined Green's function, say, g^-. For this aim, as we have already shown, relations of the type (4.19) can be very useful. In order to derive them we start again from the Hilbert identity (4.1) where $M' = M(q')$ and $M = M(q)$ and we choose $q' = \ell_\Im(k)$, $q_1 = k_\Im$, $q_2 = a^2 - 0$ (see (2.10), (4.7) and (4.16)). Then, passing to the objects with hats by (2.12), recalling definitions (4.7), (4.18) and keeping only the leading term in the neighborhood of $k \sim \pm ia$ we get

$$G(k) - G^-(k) = G(k)\overleftarrow{\mathcal{L}}_1(G_1(k) - G_1^-(k))\overrightarrow{\mathcal{L}}_1 G^-(k) + o(1), \quad k \sim \pm ia.$$

Inserting the explicit singular behaviors of $G_1(k)$, $G_1^-(k)$ and $G^-(k)$ at $k = \pm ia$ given in (3.34), (3.35) and (4.37), we have

$$G(k) - g^- = -\frac{c\theta(|k_\Im| - a)}{2 + \lambda}\phi^- \otimes \psi^- +$$
$$+ c\left(-\frac{1}{\pi}\operatorname{arccot}\frac{a - |k_\Im|}{|k_\Re|} + \frac{\theta(|k_\Im| - a)}{2}\right)G(k)\overleftarrow{\mathcal{L}}_1\Phi_{1,-a} \otimes \Psi^-(k) + o(1),$$

where in the last multiplier the definition of $\Psi^-(k)$ in (4.20) was used. Again by (4.20) and (4.37) $\Psi^-(k) = \frac{2 + \lambda\theta(a - |k_\Im|)}{2 + \lambda}\psi^-$, where as always ψ^- denotes $\Psi^-(k)$ for $|k_\Im| < a$ by (4.27). Then

$$G(k) - g^- = \left\{-\frac{c\theta(|k_\Im| - a)}{2 + \lambda}\phi^- + c\left(-\frac{1}{\pi}\operatorname{arccot}\frac{a - |k_\Im|}{|k_\Re|} + \frac{\theta(|k_\Im| - a)}{2}\right) \times \right.$$
$$\left. \times \frac{2 + \lambda\theta(a - |k_\Im|)}{2 + \lambda}G(k)\overleftarrow{\mathcal{L}}_1\Phi_{1,-a}\right\} \otimes \psi^- + o(1). \tag{5.1}$$

Thus in order to get the behavior of $G(k)$ we need to find that of $G(k)\overleftarrow{\mathcal{L}}_1\Phi_{1,-a}$, which follows by applying to (5.1) operation $\overleftarrow{\mathcal{L}}_1\Phi_{1,-a}$ from the right and using again (4.20), (4.27) and (4.29). Then

$$G(k)\overleftarrow{\mathcal{L}}_1\Phi_{1,-a} = \frac{\pi\phi^-}{A(k)} + o(1), \quad k \sim \pm ia, \tag{5.2}$$

where we denoted for shortness

$$A(k) = \pi + \lambda\operatorname{arccot}\frac{a - |k_\Im|}{|k_\Re|}. \tag{5.3}$$

This function is real, positive thanks to (4.32) and discontinuous at $k = \pm ia$. Now inserting (5.2) in (5.1) we derive finally that

$$G(k) = g^- - \frac{c}{A(k)}\left(\operatorname{arccot}\frac{a - |k_\Im|}{|k_\Re|}\right)\phi^- \otimes \psi^- + o(1), \quad k \sim \pm ia. \tag{5.4}$$

Applying to (5.4) from the left the operation $\Psi_{1,a}\overrightarrow{\mathcal{L}}_1$ and recalling the definitions (4.20) and (4.27) we derive

$$\Psi_{1,a}\overrightarrow{\mathcal{L}}_1 G(k) = \frac{\pi\psi^-}{A(k)} + o(1), \quad k \sim \pm ia, \tag{5.5}$$

and by (4.29) also

$$c\Psi_{1,a}\overrightarrow{\mathcal{L}}_1 G(k)\overleftarrow{\mathcal{L}}_1\Phi_{1,-a} = \frac{\pi\lambda}{A(k)} + o(1), \quad k \sim \pm ia. \tag{5.6}$$

Correspondingly, we get for the behavior of the Jost solutions in the neighborhood of $k = \pm ia$, thanks to (3.3) and (5.2)

$$\Phi(k) = \frac{i\pi c\phi^-}{A(k)(k - ia)} + O(1), \quad k \sim ia, \tag{5.7}$$

$$\Phi(k) = \frac{\pi\phi^-}{A(k)} + o(1), \quad k \sim -ia \tag{5.8}$$

and analogous relations for $\Psi(k)$.

Now we are ready to consider the d-bar derivative in the sense of distributions of the Jost solution, say, $\Phi(k)$. Let first $k \neq \pm ia$. Then we use (3.3), (4.6), and (4.10) to derive

$$\frac{\partial \Phi(k)}{\partial \bar{k}} = \Phi(-\bar{k}) r(k), \quad k \neq \pm ia, \tag{5.9}$$

where the Spectral data are defined as follows

$$r(k) = \frac{\operatorname{sgn} k_{\Re}}{2\pi} \left(\Psi_1(-\bar{k}) \overrightarrow{\mathcal{L}}_1 G(k) \overleftarrow{\mathcal{L}}_1 \Phi_1(k) \right). \tag{5.10}$$

Thanks to (3.3), (3.4) and (5.6) we get the singular behavior of these spectral data in the form

$$r(k) = \frac{i\lambda \operatorname{sgn} k_{\Re}}{2(k_{\Re} + i|k_{\Im}| - ia)A(k)} + o(1), \quad k \sim \pm ia, \tag{5.11}$$

i.e. in both points it has a pole singularity multiplied by the discontinuous function $A(k)$. Taking into account that the singular behavior of $\Phi(-\bar{k})$ is given by the denominator $\overline{(k - ia)}A(k)$ at point $k = ia$ and by $A(k)$ at point $k = -ia$ we see that the r.h.s. in (5.9) is integrable at the latter point but it has a singularity $\operatorname{sgn} k_{\Re} |k - ia|^{-2} A(k)^{-2}$ at point $k = ia$, that is not integrable. On the other side, $\Phi(k)$ is locally integrable for any k so $\Phi(x, k) e^{i\ell(k)x}$ is a Schwartz distribution with respect to k. Thus its d-bar derivative in the sense of distributions exists and can be defined in the standard way. Let $f(k)$ be a test function that properly decays at infinity (we are not interested in the exponential growth due to the multiplier $e^{-i\ell(k)x}$ now). Then the d-bar derivative of $\Phi(k)$ is defined as

$$\int d^2k \, \frac{\partial \Phi(k)}{\partial \bar{k}} f(k) = -\int d^2k \, \Phi(k) \frac{\partial f(k)}{\partial \bar{k}} = -\lim_{\varepsilon \to 0} \int_{|k \pm ia| > \varepsilon} d^2k \, \Phi(k) \frac{\partial f(k)}{\partial \bar{k}},$$

where in the last equality we again used the property of local integrability of $\Phi(k)$. Integrating by parts for $\varepsilon > 0$ we can use (5.9) and we have

$$-\lim_{\varepsilon \to 0} \int_{|k \pm ia| > \varepsilon} d^2k \, \Phi(k) \frac{\partial f(k)}{\partial \bar{k}} =$$

$$= \frac{f(ia)}{2i} \lim_{\varepsilon \to 0} \oint_{|k - ia| = \varepsilon} dk \, \Phi(k) + \lim_{\varepsilon \to 0} \int_{|k - ia| > \varepsilon} d^2k \, \Phi(-\bar{k}) r(k) f(k),$$

where we omitted the term $\oint_{|k + ia| > \varepsilon}$ since thanks to (5.8) it gives zero in the limit $\varepsilon \to 0$. Thanks to (5.7) and (5.8) both limits in the r.h.s. exist. To be more precise let us introduce the distribution

$$\text{p.v.} \int d^2k \frac{\operatorname{sgn} k_{\Re} f(k)}{|k - ia|^2 A(k)^2} = \lim_{\varepsilon \to 0} \int_{|k - ia| > \varepsilon} d^2k \frac{\operatorname{sgn} k_{\Re} f(k)}{|k - ia|^2 A(k)^2}. \tag{5.12}$$

Notice the presence in the numerator of $\operatorname{sgn} k_{\Re}$ that guaranties existence of the limit. We used the principal value (p.v.) notation in analogy with the one-dimensional case. It can be checked directly that

$$\text{p.v.} \int d^2k \frac{\operatorname{sgn} k_{\Re} f(k)}{|k - ia|^2 A(k)^2} = \int d^2k \, \frac{\operatorname{sgn} k_{\Re}}{|k - ia|^2 A(k)^2} \left[f(k) - \theta(\delta - |k - ia|) f(ia) \right] =$$

$$= \frac{1}{2} \int d^2k \, \operatorname{sgn} k_{\Re} \frac{f(k) - f(-\bar{k})}{|k - ia|^2 A(k)^2}, \tag{5.13}$$

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where δ is some real positive parameter and the second term in (5.13) is independent on the choice of δ. In the case where a distribution has singularities of this form at some finite number of points a_1, a_2, etc., we use the same notation for the integral assuming that either the cutoff procedure in (5.12) or the subtraction procedure in (5.13) is performed at each point. Of course, the parameters ε_j and δ_j must be chosen in such way that corresponding discs do not overlap.

Let us denote

$$\Phi_a = -\frac{1}{2\pi} \lim_{\varepsilon \to 0} \oint_{|k-ia|=\varepsilon} dk\, \Phi(k), \tag{5.14}$$

so that $i\Phi_a$ can be considered as an extension of the definition of residuum to the case in which the pole singularity is multiplied by a function discontinuous at the same point. Thanks to (5.7) we get that this limit also exists and equals

$$\Phi_a = c\frac{\log(1+\lambda)}{\lambda}\,\phi^-. \tag{5.15}$$

Thus, summarizing all above definitions we get that

$$\frac{\partial \Phi(k)}{\partial \bar{k}} = \Phi(-\bar{k})r(k) + i\pi\Phi_a\delta(k-ia), \tag{5.16}$$

where $\Phi(x,-\bar{k})r(k)$ is now a distribution in k defined by the p.v. prescription given above. By (5.7), (5.8), and (5.11) it is integrable at $k = -ia$, but it behaves as $|k - ia|^{-2}A^{-2}(k)$ in the neighborhood of the point $k = ia$.

Equation (5.16) supplies us with the first equation of the Inverse problem. In order to close it we need the analog of the first relation in (3.17), where it is stated that the residuum of the function is proportional to its value in the conjugated point. But in our case $\Phi(k)$ is discontinuous at point $k = -ia$, so again some modification of the notion of "value" at this point must be given. Following the procedure used in (5.14) we can define it as

$$\Phi_{-a} = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \oint_{|k+ia|=\varepsilon} \frac{dk}{k+ia}\, \Phi(k). \tag{5.17}$$

Thanks to (5.8) this limit also exists and equals

$$\Phi_{-a} = \frac{\log(1+\lambda)}{\lambda}\,\phi^-, \tag{5.18}$$

so that by (5.15) we have

$$\Phi_a = c\Phi_{-a}, \tag{5.19}$$

that shows that the parameter c is not modified by the perturbation. This equality closes the formulation of the Inverse problem (5.16). Finally, taking into account the asymptotic behavior of $\Phi(x,k)$ and (5.17) and (5.19) we can formulate the Inverse problem as the following system of integral equations:

$$\Phi(x,k) = e^{-i\ell(k)x} +$$

$$+ \frac{1}{\pi}\text{p.v.} \int \frac{d^2k'}{k-k'}\, e^{i(\ell(k')-\ell(k))x}\Phi(x,-\overline{k'})r(k') + i\frac{e^{i(\ell(ia)-\ell(k))x}}{k-ia}\,\Phi_a(x), \tag{5.20}$$

$$\frac{1}{c}\Phi_a(x) = e^{-i\ell(-ia)x} -$$

$$- \frac{1}{\pi}\text{p.v.} \int d^2k\, \frac{\Phi(x,-\overline{k})r(k)}{k+ia}e^{i(\ell(k)-\ell(-ia))x} + \frac{e^{i(\ell(ia)-\ell(-ia))x}}{2a}\,\Phi_a(x). \tag{5.21}$$

The integrands in the r.h.s. of these two equations are not locally integrable, respectively, the first at $k = ia$ and the second at $k = \pm ia$. Correspondingly, their integrals are regularized by means of the principal value prescription, as in (5.12) or (5.13), at $k = ia$ and at $k = \pm ia$.

The potential is reconstructed by means of

$$u(x) = -\frac{2i}{\pi}\text{p.v.} \int d^2k \, \frac{\partial}{\partial x_1}\Big(e^{i\ell(k)x}\Phi(x, -\overline{k})r(k) \Big) + 2\frac{\partial}{\partial x_1}\Big(e^{i\ell(ia)x}\,\Phi_a(x) \Big). \qquad (5.22)$$

6   Conclusion

In this article on the basis of the resolvent approach we gave a detailed presentation of an extension of the inverse scattering theory for the heat operator to the case where the potential (1.5) is a perturbation of the 1-dimensional one soliton potential $u_1(x_1)$ (3.1) by means of a smooth, decaying at infinity function $u_2(x)$ of two space variables. To our knowledge this is the first time that inverse scattering theory is applied to a non-scattering situation, i.e. a situation with an infinite obstacle. As a result of our investigation we proved that under such a perturbation the Jost solutions get specific singularities (5.7) and (5.8) on the complex plane of the spectral parameter k. We demonstrated that the d-bar problem (5.16), (5.19), while looking familiar for a potential whose spectrum has a discrete and continuous part, needs a substantially modified approach due to the singularity structure of the spectral data given in (5.11). It was necessary to establish the meaning in the sense of distributions of all terms involved in this problem, in order to be able to formulate the Inverse problem as the system of integral equations (5.20), (5.21). It is easy to check that the singular behavior of the spectral data and Jost solution as given in (5.11) and (5.7), (5.8) is compatible with this Inverse problem. On the other side, it is necessary to prove that the potential $u(x)$ reconstructed by means of (5.22) is of the type (1.5). We plan to address this problem in a forthcoming work.

Another open problem is the application of these results to the KPII equation (1.3) itself. In particular, investigation of the time asymptotics of solutions with initial data of the type (1.5) must be performed. Let us mention only that the singular behavior (5.11) of the spectral data is preserved under evolution (1.3). Indeed [3], the time dependence of the spectral data is given as

$$r(k, t) = e^{4i(k^3 + \bar{k}^3)}. \qquad (6.1)$$

Thus we get that

$$a = \text{const}, \qquad \lambda = \text{const} \qquad (6.2)$$

also with respect to time.

In Sec. 3 we mentioned that the above construction can be easily generalized to the case where the potential $u_1(x_1)$ is a pure N-soliton 1-dimensional potential. At the same time our approach also admits straightforward generalization to the case where $u_1(x)$ is not a function of one space variable but the result of application of the Bäcklund transformation to a generic background 2-dimensional potential $u_0(x)$ decaying on the x-plane. Then the inverse problem is again given by Eqs. (5.20) and (5.21), where the spectral data $r(k)$ are replaced with

$$r(k) + \frac{(k + ia)(\bar{k} + ia)}{(\bar{k} - ia)(k - ia)}r_0(k), \qquad (6.3)$$

where $r(k)$ is of the type (5.11) and $r_0(k)$ are the spectral data of the potential $u_0(x)$ (see [11]).

The theory of the heat equation with respect to the nonstationary Schrödinger equation is in some respects simpler and in some other respects unexpectedly more difficult. As we have shown, under perturbation the Jost solution get singularities more complicated than poles, but this solution has no additional cut in the complex domain, in contrast with the nonstationary Schrödinger case as discovered in [13]. On the other side the generalization of this scheme to the case of multi-ray structure of the potential $u(x)$ meets with essential problems, first of all due to the fact that the resolvent (or Green's function) of the heat equation even of a 2-soliton (generic) potential is unknown in the literature. This problem also needs future development.

Acknowledgment

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